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Exact Results for Conditional Means of a Passive Scalar in Certain Statistically Homogeneous Flows

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For a passive scalar $T(\mathbf{r}, t)$ randomly advected by a statistically homogeneous flow, the probability density function (pdf) of its fluctuation can in general be expressed in terms of two conditional means: $\langle \nabla^2 T \mid T \rangle$ and $\langle |\nabla T|^2 \mid T \rangle$. We find that in some special cases, either one of the two conditional means can be obtained explicitly from the equation of motion. In the cases when there is no external source and that the normalized fluctuation reaches a steady state or when a steady state results from a negative damping, $\langle \nabla^2 T \mid T \rangle = -(\langle |\nabla T|^2 \rangle / \langle T^2 \rangle)\, T$. The linearity of the conditional mean in these cases follows directly from the fact that the advection equation of a passive scalar is linear. On the other hand, when the scalar is supported by a homogeneous white-in-time external source, $\langle |\nabla T|^2 \mid T \rangle = \langle |\nabla T|^2 \rangle$.

KEY WORDS: Passive scalar; conditional means; statistically homogeneous flows.

I. INTRODUCTION

In turbulent flows, physical fields such as velocity, temperature and pressure display irregular and complex fluctuations both in time and in space. This feature renders the need for a probabilistic description of turbulence. A key issue is thus to understand the statistics of various physical quantities in fully developed turbulent flows. For example, one would like to understand the behavior of velocity differences between two-points separated by a distance when the separation is within the inertial range. The inertial range refers to the range of length scales that are smaller than those of energy input and larger than those affected directly by molecular dissipation. This problem was first addressed by Kolmogorov more than

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fifty years ago,[1] and a main result is that the velocity differences have simple scaling within the inertial range. An interesting question is then how the simple scaling behavior can be corrected, as indicated by various experimental measurements. Such correction is known as anomalous scaling. A similar question concerning the scaling of a randomly adverted scalar is a theoretically more tractable problem and has attracted much attention recently.[2–14] Apart from the question of whether there exists any anomalous scaling, the general phenomenon of non-Gaussianity of scalar fluctuations[15–22] remains imperfectly understood.

The advection of a scalar field $T(\mathbf{r}, t)$, for example, the temperature field, by a random velocity field $\mathbf{u}(\mathbf{r}, t)$ is described by the following equation:

$$\frac{\partial T}{\partial t} + \mathbf{u} \cdot \nabla T = \kappa \nabla^2 T + s \tag{1}$$

where κ is the molecular diffusivity of the scalar T and $s(\mathbf{r}, t)$ is an external source driving the scalar. The velocity field $\mathbf{u}(\mathbf{r}, t)$ is taken to be incompressible,

$$\nabla \cdot \mathbf{u} = 0 \tag{2}$$

and has prescribed statistics. We are interested in the statistics of the passive scalar T, and we shall consider only statistically homogeneous flows in the present work.

It has been shown that[23] the probability density function (pdf) of any stationary fluctuation can be expressed in terms of two conditional means of the time derivatives of the fluctuation. An analogous result applies to any statistically homogeneous fluctuation such that its pdf is related to two conditional means of its spatial gradients.[24] Thus the problem of understanding the statistics of any Statistically homogeneous fluctuation is equivalent to the problem of understanding the two conditional means. In general, the calculation of the conditional means directly from the equation of motion is a very difficult problem. In this paper, we shall show that exact results for either one of the conditional means can be obtained explicitly in some special cases. Since only one of the two can be found, the pdf is still undetermined up to the remaining conditional mean.

II. PROBABILITY DENSITY FUNCTION AND THE CONDITIONAL MEANS

For completeness, we shall outline how the pdf of the passive temperature in a statistically homogeneous flow can be related to two conditional means of its spatial gradients, and the extension of this result for temperature differences.

Suppose the temperature field $T(\mathbf{r}, t)$ is measured as a function of position $\mathbf{r}$ at a certain time t in a statistically homogeneous turbulent fluid flow. Because of homogeneity, we have

$$\langle \nabla \cdot [g(T)\, \nabla T] \rangle = 0 \tag{3}$$

for any well-behaved function $g(T)$ of T, where $\langle \cdots \rangle$ denotes an ensemble average. As a result,

$$\langle g(T)\, \nabla^2 T \rangle = -\langle g'(T)\, |\nabla T|^2 \rangle \tag{4}$$

where $g'(T) \equiv dg(T)/dT$.

We write ensemble averages in terms of integrals of the pdf of T, $P(T)$. After integrating by parts once, (4) becomes

$$\int g(T) \left\{ \langle \nabla^2 T \mid T \rangle\, P(T) - \frac{d}{dT} [\langle |\nabla T|^2 \mid T \rangle\, P(T)] \right\} dT = 0 \tag{5}$$

where $\langle \nabla^2 T \mid T \rangle$ is the conditional mean of the Laplacian of the temperature field when the temperature fluctuation is at a certain value, and is thus a function of T. The conditional mean $\langle |\nabla T|^2 \mid T \rangle$ is similarly defined. Since (5) holds for any well-behaved $g(T)$, the expression inside the curly bracket has to be identically zero:

$$\langle \nabla^2 T \mid T \rangle\, P(T) - \frac{d}{dT} [\langle |\nabla T|^2 \mid T \rangle\, P(T)] = 0 \tag{6}$$

Solving this differential equation for $P(T)$, we find

$$P(T) = \frac{C_N}{\langle |\nabla T|^2 \mid T \rangle} \exp\left(\int_0^T \frac{\langle \nabla^2 T \mid T' \rangle}{\langle |\nabla T|^2 \mid T' \rangle}\, dT' \right) \tag{7}$$

where C_N is the normalization constant determined by $\int P(T)\, dT = 1$. Thus the pdf $P(T)$ can be expressed in terms of the two conditional means, $\langle |\nabla T|^2 \mid T \rangle$ and $\langle (\nabla^2 T) \mid T \rangle$. We can calculate any one of these three functions of T if the other two are known.

Identities like (6) and (7) relating the pdf and conditional means of the simultaneous temperature difference $\Delta T(\mathbf{r}, \mathbf{r}', t) \equiv T(\mathbf{r}, t) - T(\mathbf{r}', t)$ can be similarly obtained. Instead of (3), the starting point is

$$\langle (\nabla_r + \nabla_{r'}) \cdot [g(\Delta T)(\nabla_r + \nabla_{r'})\, \Delta T] \rangle = 0 \tag{8}$$

The manipulations corresponding to (4)–(7), together with a use of $\nabla_r \cdot \nabla_{r'} \Delta T \equiv 0$, yield

$$\langle (\nabla_r^2 + \nabla_{r'}^2)\, \Delta T \mid \Delta T \rangle\, P_\Delta(\Delta T) - \frac{d}{d\Delta T} [\langle |(\nabla_r + \nabla_{r'})\, \Delta T|^2 \mid \Delta T \rangle\, P_\Delta(\Delta T)] = 0 \tag{9}$$

and

$$P_\Delta(\Delta T) = \frac{C_\Delta}{\langle |(\nabla_r + \nabla_{r'})\, \Delta T|^2 \mid \Delta T \rangle} \exp\left(\int_0^T \frac{\langle (\nabla_r^2 + \nabla_{r'}^2)\, \Delta T \mid \Delta T' \rangle}{\langle |(\nabla_r + \nabla_{r'})\, \Delta T|^2 \mid \Delta T' \rangle}\, d\Delta T' \right) \tag{10}$$

where P_Δ is the pdf of ΔT.

In Sections III and IV, we shall consider two special cases, one in which there is no heat source and the other in which the heat source is homogeneous and white-in-time, and obtain exact results for the conditional means.

III. NO HEAT SOURCE

We first consider the case when there is no heat source: $s = 0$ in (1). Multiplying (1) by $2nT^{2n-1}$ and taking an ensemble or spatial average, we have

$$\frac{d}{dt} \langle T^{2n} \rangle + \langle \mathbf{u} \cdot \nabla (T^{2n}) \rangle = 2n\kappa \langle T^{2n-1}\, \nabla^2 T \rangle \tag{11}$$

A partial space integration shows that the second term on the left hand side vanishes by homogeneity and (2). The average on the right hand side can also be transformed by a partial space integration to yield

$$\frac{d}{dt} \langle T^{2n} \rangle = -2n(2n-1)\, \kappa \langle T^{2n-2}\, |\nabla T|^2 \rangle \tag{12}$$

The right hand side of (12) is negative, expressing the decay of temperature fluctuations. The decay rate for the temperature variance is given by $2\kappa \langle |\nabla T|^2 \rangle / \langle T^2 \rangle$, which is the ratio of the scalar dissipation to the variance.

Now suppose the normalized fluctuation, $X \equiv T/T_{\text{rms}}$, reaches a statistically stationary state although T itself decays in time.[17] Here, T_{rms} is the root-mean-square of T. As a result,

$$\frac{d\langle X^{2n}\rangle}{dt} = 0 \tag{13}$$

which relates the decay rate of the nth moment, $\langle T^{2n}\rangle$, to that of the variance, $\langle T^2\rangle$. The approach to stationary X involves some delicate questions. Of course stationarity requires that the velocity field be statistically stationary. Suppose that the flow volume is infinite, that the velocity field spectrum and initial scalar spectrum have a low-cutoff wavenumber k_c. The advection process will induce a tail $k \ll k_c$ in the scalar spectrum, and this tail will forever move to ever-lower k. If, instead, the flow is contained in a cyclic box, there is a gravest wavenumber k_0. In this case, a stationary cascade is expected when the low-k tail has saturated at k_0.

Equation (13) yields

$$\langle T^{2n-1}\,\nabla^2 T\rangle = -\frac{\langle|\nabla T|^2\rangle}{\langle T^2\rangle}\langle T^{2n}\rangle \tag{14}$$

which is true for arbitrary n, and thus implies

$$\langle \nabla^2 T \mid T\rangle = -\frac{\langle|\nabla T|^2\rangle}{\langle T^2\rangle}\, T \tag{15}$$

Equation (15) holds if there is no external source and that the scalar fluctuation decays in such a fashion that its normalized fluctuation is stationary. Then $P(T)$ is determined by $\langle |\nabla T|^2 \mid T\rangle$ only, and it is given by

$$P(T) = \frac{C_N}{\langle |\nabla T|^2 \mid T\rangle} \exp\left[-\frac{\langle|\nabla T|^2\rangle}{\langle T^2\rangle}\int_0^T \frac{T'\,dT'}{\langle |\nabla T|^2 \mid T'\rangle}\right] \tag{16}$$

This result for the pdf was derived by Sinai and Yakhot.[13] They did not emphasize the implied linearity (15) of the conditional mean.

Equation (15) also holds in the statistically steady state produced if the source s has the form of a suitable negative damping term:

$$s = \frac{aT}{f(T_{\text{rms}}/T_0)} \tag{17}$$

Here a is a positive growth rate, T_0 is a chosen reference level, and f is a monotonically growing function of argument satisfying $f(0)=1$. The nonlinearity provided by f ensures that a steady state exists in which

$$\frac{a\langle T^2\rangle}{f(T_{\mathrm{rms}}/T_0)}=\kappa\langle|\nabla T|^2\rangle \tag{18}$$

As before, the advection term drops out of the balance. A repeat of the operations that lead from (14) to (15), and use of (18), yield (15) for the present case also.

In the case when (15) holds, whether the pdf $P(T)$ is Gaussian or not depends on whether the conditional mean $\langle|\nabla T|^2\mid T\rangle$ is independent of T or not. The non-Gaussianity of scalar fluctuations is thus a direct consequence of a non-zero correlation between $|\nabla T|^2$ and T.

The linearity of (15) is a consequence of the fact that the advection equation for the passive scalar is linear. The conditional mean no longer is linear if nonlinearity is introduced.

Suppose there is an additional nonlinear term $h(T)$ on the right hand side of (1) but no source. The stationarity of the normalized field now requires

$$\langle[\kappa\,\nabla^2 T+h(T)]\,T^{2n-1}\rangle=\frac{\langle[\kappa\,\nabla^2 T+h(T)]\,T\rangle}{\langle T^2\rangle}\langle T^{2n}\rangle \tag{19}$$

which implies that the conditional mean $\langle\nabla^2 T\mid T\rangle$ is a non-linear function of T:

$$\kappa\langle\nabla^2 T\mid T\rangle=\left[\frac{\kappa\langle T\,\nabla^2 T\rangle+\langle h(T)\,T\rangle}{\langle T^2\rangle}\right]T-h(T) \tag{20}$$

IV. HOMOGENEOUS AND WHITE-IN-TIME HEAT SOURCE

Next, we consider the case where the heat source s is a homogeneous white-in-time field that satisfies

$$\langle s(\mathbf{r},t)\rangle=0,\qquad\langle s(\mathbf{r},t)\,s(\mathbf{r},t')\rangle=2q\delta(t-t') \tag{21}$$

Multiplying both sides of (1) by a well-behaved function $g(T)$ of T and taking (ensemble) spatial averages, we get the steady-state balance equation

$$\kappa\langle g(T)\,\nabla^2 T\rangle+\langle sg(T)\rangle=0 \tag{22}$$

where $G'(T) = g(T)$. Here we have taken $d\langle G(T)\rangle/dt = 0$ and $\langle \mathbf{u} \cdot \nabla G(T)\rangle = 0$. The later relation follows because of homogeneity and incompressibility as in the previous case.

To evaluate the term $\langle sg(T)\rangle$, we invoke the equation of motion of $g(T)$:

$$\frac{\partial g(T)}{\partial T} = -\mathbf{u} \cdot \nabla g(T) + \kappa g'(T)\, \nabla^2 T + sg'(T) \tag{23}$$

Integrating this equation from t' to t with $t > t'$, then multiplying $s(\mathbf{r}, t)$ and taking ensemble average, we find

$$\langle s(\mathbf{r}, t)\, g(T(\mathbf{r}, t))\rangle = \int_{t'}^{t} \langle s(\mathbf{r}, t)\, s(\mathbf{r}, \tilde{t})\, g'(T(\mathbf{r}, \tilde{t}))\rangle\, d\tilde{t} \tag{24}$$

In obtaining (24), we have used the fact that $T(\mathbf{r}, \tilde{t})$ is independent of $s(\mathbf{r}, t)$ for $\tilde{t} < t$. Let t' tend to t and use the property of the source s (21). We then get

$$\langle sg(T)\rangle = q\langle g'(T)\rangle \tag{25}$$

Using (22) and (25) together with (4), we find

$$\langle g'(T)\, |\nabla T|^2\rangle = \frac{q}{\kappa} \langle g'(T)\rangle \tag{26}$$

Equation (26) shows that the square of the temperature gradient is uncorrelated with any well-behaved function of the temperature field. Such feature was also found for the vorticity gradient and the vorticity field in two-dimensional turbulence driven by a white-in-time force.[25] Writing (26) in terms of integrals of $P(T)$ and noting that it is valid for any arbitrary well-behaved function $g'(T)$, we finally get

$$\langle |\nabla T|^2 \mid T\rangle = \langle |\nabla T|^2\rangle = \frac{q}{\kappa} \tag{27}$$

Equations (27) and (6) imply that $\langle \nabla^2 T \mid T\rangle$ is positive when T is negative and negative when T is positive if $P(T)$ is a decreasing function of $|T|$. Also, (7) reduces to

$$P(T) = C_N \exp\left[\frac{\kappa}{q} \int_0^T \langle \nabla^2 T \mid T'\rangle\, dT'\right] \tag{28}$$

V. SUMMARY

We have shown that in some special cases, exact results for one of the two conditional means, $\langle \nabla^2 T \mid T \rangle$ and $\langle |\nabla T|^2 \mid T \rangle$, can be obtained directly from the equation of motion. The results are summarized as follows. In the cases where there is no heat source and the normalized fluctuation reaches a steady state, or where an absolute steady state results from a negative damping, we find that $\langle \nabla^2 T \mid T \rangle$ is proportional to T and is given by $-(\langle |\nabla T|^2 \rangle / \langle T^2 \rangle)\, T$. It should be emphasized that this linearity result is independent of the statistics of the homogeneous velocity field. On the other hand, if there is a homogeneous white-in-time heat source, $\langle |\nabla T|^2 \mid T \rangle$ is independent of T and is equal to q/κ, with q being the autocorrelation of the source and κ is the molecular diffusivity of the scalar.

In neither case can the results be extended to means conditional on the temperature difference ΔT across a finite spatial distance. This is because the advection terms in the equations of motion for moments of ΔT cannot be eliminated by partial space integration.

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REFERENCES

1. A. N. Kolmogorov, *Dokl. Akad. Nauk. SSSR* **31**:538 (1941).
2. R. H. Kraichnan, *Phys. Rev. Lett.* **72**:1016 (1994).
3. V. S. L'vov, I. Procaccia, and A. L. Fairhall, *Phys. Rev. E* **50**:4684 (1994).
4. B. I. Shraiman and E. D. Siggia, *Phys. Rev. E* **49**:2912 (1994); *Comptes Rendus de l'Academie des Sciences, Serie II* **321**:279 (1995); *Phys. Rev. Lett.* **77**:2463 (1996).
5. R. H. Kraichnan, V. Yakhot, and S. Chen, *Phys. Rev. Lett.* **75**:240 (1995).
6. K. Gawedzki and A. Kupiainen, *Phys. Rev. Lett.* **75**:3608 (1995).
7. M. Chertkov, G. Falkovich, I. Kolokolov, and V. Lebedev, *Phys. Rev. E* **52**:4924 (1995).
8. M. Chertkov and G. Falkovich, *Phys. Rev. Lett.* **76**:2706 (1996).
9. M. Chertkov, G. Falkovich, and V. Lebedev, *Phys. Rev. Lett.* **76**:3703 (1996).
10. A. L. Fairhall, O. Gat, V. L'vov, and I. Procaccia, *Phys. Rev. E* **53**:3518 (1996).
11. D. Bernard, K. Gawedzki, and A. Kupiainen, *Phys. Rev. E* **54**:2564 (1996).
12. R. H. Kraichnan, *Phys. Rev. Lett.* **78**:4922 (1997).
13. A. L. Fairhall, B. Galanti, V. S. L'vov, and I. Procaccia, *Phys. Rev. Lett.* **79**:4166 (1997).
14. E. S. C. Ching, *Phys. Rev. Lett.* **79**:3644 (1997).
15. F. Heslot, B. Castaing, and A. Libchaber, *Phys. Rev. A* **36**:5870 (1987).

16. B. Castaing, G. Gunaratne, F. Heslot, L. P. Kadanoff, A. Libehaber, S. Thomae, X.-Z. Wu, S. Zaleski, and G. Zanetti, *J. Fluid Mech.* **204**:1 (1989).
17. Y. G. Sinai and V. Yakhot, *Phys. Rev. Lett.* **63**:1962 (1989).
18. H. Chen, S. Chen, and R. H. Kraichnan, *Phys. Rev. Lett.* **63**:2657 (1989).
19. A. Pumir, B. I. Shraiman, and E. D. Siggia, *Phys. Rev. Lett.* **66**:2984 (1991).
20. M. Holzer and A. Pumir, *Phys. Rev. E* **47**:202 (1993).
21. E. S. C. Ching and Y. Tu, *Phys. Rev. E* **49**:1278 (1994).
22. M. Chertkov, G. Falkvoich, I. Kolokolov, and V. Lebedev, *Phys. Rev. E* **51**:5609 (1995).
23. S. B. Pope and E. S. C. Ching, *Phys. Fluids A* **5**:1529 (1993).
24. E. S. C. Ching, *Phys. Rev. E* **53**:5899 (1996); *Phys. Rev. E* **55**:4830 (1997).
25. G. Eyink, *Physica D* **91**:97 (1996).